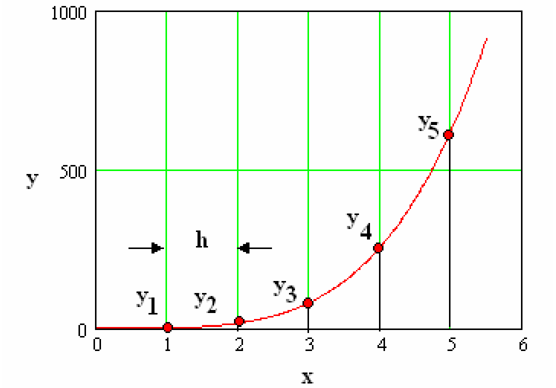
ENGINEERING COUNCIL EXAM MATHEMATICS C101
Solutions to exam paper 2004

Q2a. Estimate the value of the definite integral $I = \int_1^5 x^4 dx$ by Simpson's rule using four strips. What is the error in the estimate? How many strips would be needed to guarantee that the estimate was correct to 3 decimal places?

(The error, e_s, in Simpson's rule using n strips applied to the integral $I = \int_a^b f(x)dx$ is given by

$|e_s| \le \frac{(b-a)^5}{180n^4} M$ where M is the maximum value of $\left|f^{(4)}(x)\right|$ in the interval $a \le x \le b$)

The plot shown is not really required in the exam but is included here for tutorial purposes.



Evaluate y_1 to y_5 $y_1 = 1^4 = 1$ $y_2 = 2^4 = 64$ $y_3 = 3^4 = 81$ $y_4 = 4^4 = 256$ $y_5 = 5^4 = 625$

It is easy to check the correct answer by integrating and evaluating $I = \int_1^5 x^4 dx = \left[\frac{x^5}{5}\right]_1^5 = 625$

Simpson's Rule $I = \frac{h}{3}[(\text{first} + \text{last}) + 4(\text{sum of the even ordinates}) + 2(\text{sum of the remaining odd ordinates})]$

$$I = \frac{h}{3}[(y_1 + y_5) + 4(y_2 + y_4) + 2(y_3)] = \frac{1}{3}[(1 + 625) + 4(64 + 256) + 2(81)] = 689.333$$

Error = 689.333-625 = 64.333

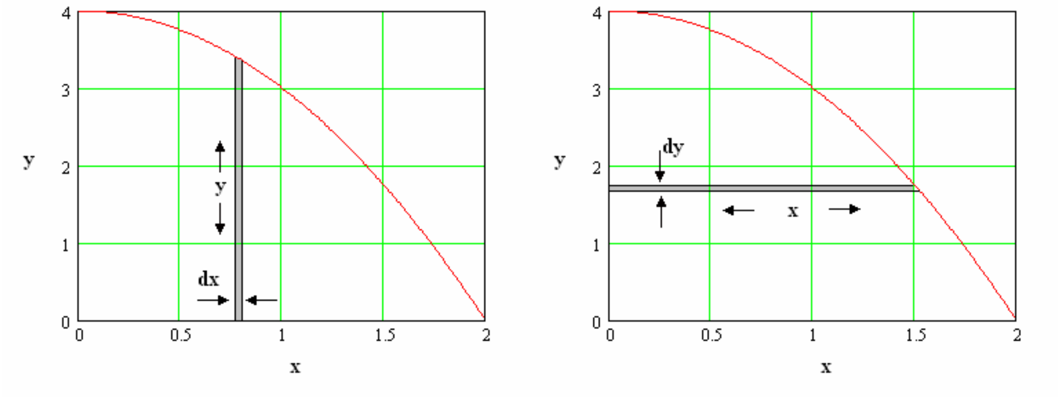
$|e_s| \le \frac{(b-a)^5}{180n^4} M$

$64.333 \le \frac{(5-1)^5}{180n^4} 625^4$ hence $n^4 = 13.493 \times 10^9$ n = 340.8 so we need 342 strips to have an even number large enough.

Q2(b) Find the position of the centroid of the plane area bounded by the x – axis , the y – axis and the curve $y = 4 - x^2$, $0 \le x \le 2$. What is the position of the centroid of the plane area bounded by the x–axis and the curve $y = 4 - x^2$, $-2 \le x \le 2$?

SOLUTION

The plot of $y = 4 - x^2$ is shown.



First find the area $A = \int_0^2 y\,dx = \int_0^2 (4 - x^2)dx = \left[4x - \frac{x^3}{3}\right]_0^2 = \left(8 - \frac{8}{3}\right) - 0 = 5.333$

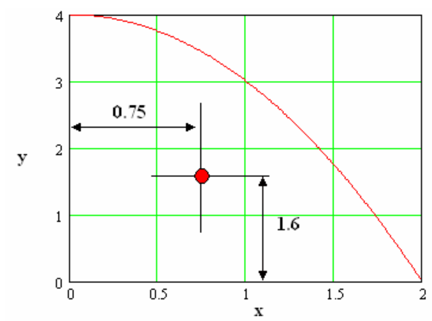
Next find the moment of area about the x axis.

$$Mx = \int_0^2 \frac{y^2}{2}dx = \int_0^2 \frac{(4 - x^2)^2}{2}dx = \int_0^2 \frac{(16 + x^4 - 8x^2)}{2}dx = \left[\frac{16x + \frac{x^5}{5} - \frac{8x^3}{3}}{2}\right]_0^2$$

$$Mx = \left[8x + \frac{x^5}{10} - \frac{4x^3}{3}\right]_0^2 = (16 + 3.2 - 10.667) - 0 = 8.533$$

The position of the centroid from the x axis is Mx/A = 8.533/5.333 = 1.6

Next find the moment of area about the y axis.



$$My = \int_0^2 yx\,dx = \int_0^2 (4 - x^2)x\,dx = \int_0^2 (4x - x^3)dx = \left[2x^2 - \frac{x^4}{4}\right]_0^2$$

$$My = (8 - 4) - 0 = 4$$

The position of the centroid from the y axis is
My/A = 4/5.333 = 0.75

For the limits of -2 to +2 the centroid moves onto the y axis at x = 0 and y = 1.6