UNIT 1: ANALYTICAL METHODS FOR ENGINEERS

Unit code: **A/601/1401**

QCF Level: **4**

Credit value: **15**

OUTCOME 1

TUTORIAL 2

EXPONENTIAL, TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Unit content

1 Be able to analyse and model engineering situations and solve problems using algebraic methods

Algebraic methods: polynomial division; quotients and remainders; use of factor and remainder theorem; rules of order for partial fractions (including linear, repeated and quadratic factors); reduction of algebraic fractions to partial fractions.

Exponential, trigonometric and hyperbolic functions: the nature of algebraic functions; relationship between exponential and logarithmic functions; reduction of exponential laws to linear form; solution of equations involving exponential and logarithmic expressions; relationship between trigonometric and hyperbolic identities; solution of equations involving hyperbolic functions.

Arithmetic and geometric: notation for sequences; arithmetic and geometric progressions; the limit of a sequence; sigma notation; the sum of a series; arithmetic and geometric series; Pascal's triangle and the binomial theorem.

Power series: expressing variables as power series functions and use series to find approximate values e.g. exponential series, Maclaurin's series, binomial series.

You should judge your progress by completing the self assessment exercises.

1. LOGARITHMIC FUNCTIONS

Equations containing logarithmic terms can be manipulated algebraically using the basic relationships that you should have learned at the national level. These are:

$a = b^n$ then $n = \log_b(a)$ where b is the base of the logarithm.

$\log_a(AB) = \log_a(A) + \log_a(B)$

$\log_a(A/B) = \log_a(A) - \log_a(B)$

$\log_a(A^n) = n\log_a(A)$

$\log_b(A) = \log_a(A)/\log_a(b)$

The following are some examples of what you need to achieve.

WORKED EXAMPLE No. 1

When a gas is compressed it is found that the pressure and volume are related by the law:

$$pV^n = C$$

Make n the subject of the formula. If $p = 200 \times 10^3$ Pascals , $V = 0.002\ m^3$ and $C = 33.3$ find n.

SOLUTIONS

$pV^n = C$ $\quad V^n = C/p$ $\quad n \log V = \log(C/p)$

$$n = \frac{\log(C/p)}{\log(V)} = \frac{\log(C) - \log(p)}{\log(V)} = \frac{\log(33.3) - \log(200 \times 10^3)}{\log(0.002)} = 1.4$$

WORKED EXAMPLE No. 2

The pressure loss in a pipe p_L is related to the flow rate Q by the equation $p_L = KQ^n$
Make n the subject of the formula. If $p_L = 150$ Pascals , $Q = 0.02\ m^3/s$ and $K = 171.5 \times 10^3$ find n.

SOLUTIONS

$p_L = KQ^n$ $\quad Q^n = p_L/K$ $\quad n \log Q = \log(p_L/K)$

$$n = \frac{\log(p_L/K)}{\log(Q)} = \frac{\log(p_L) - \log(K)}{\log(Q)} = \frac{\log(150) - \log(171.5 \times 10^3)}{\log(0.02)} = 1.8$$

2. EXPONENTIAL FUNCTIONS

You don't have to study the theory behind this but it helps to have a full understanding in order to use it. The following explanation is offered but you can jump straight to the next page if you want to.

Suppose you had a function of y = f(x) such that the gradient of the function at any point is equal to the value of the function.

In other words we want a function such that y = f(x) and $\frac{dy}{dx} = y$

Such a function is a series $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ note ! means factorial

If we differentiate this we get:

$\frac{dy}{dx} = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} .... = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ....$ Hence $\frac{dy}{dx} = y$

Note that if we differentiate y^a with respect to x we get $\frac{d(y^a)}{dx} = ay^{a-1}\frac{dy}{dx}$

Since $\frac{dy}{dx} = y$ it follows that $\frac{d(y^a)}{dx} = ay^{a-1}y = ay^a$

Now consider the function $y_1 = 1 + z + \frac{(z)^2}{2!} + \frac{(z)^3}{3!} + ...$

$\frac{dy_1}{dz} = 1 + z + \frac{z^2}{2!} + \frac{3z^3}{3!} .... = y_1$ Let z = ax $\frac{dy_1}{d(ax)} = y_1$ $\frac{dy_1}{dx} = ay_1$

If we compare $\frac{dy_1}{dx} = ay_1$ and $\frac{d(y^a)}{dx} = ay^a$ we see that if $y^a = y_1$ they are the same. Making this the case it follows that $y_1 = y^a = 1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} ....$

If we put a = 1/x then $y^{1/x} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} ....$

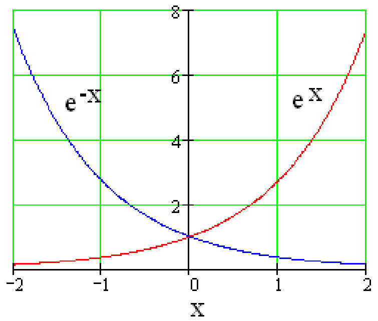
This series may be evaluated. It is not exact but to 3 significant figures it is 2.718.This number is a very important number and it is denoted 'e' and called the exponential.

$y^{1/x} = e$ and so $y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ....$

In summary we have shown the following: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ....$ e = 2.718

If $y = e^x$ then $\frac{dy}{dx} = e^x$ and extending this if $y = e^{ax}$ then $\frac{dy}{dx} = ae^{ax}$

Integration being the reverse process it follows that $\int e^{ax} = \frac{e^{ax}}{a} + C$



Plot of e^x and e^{-x}

3. NATURAL LOGARITHMS

If we use logarithms to the base of e we now see that $y = \log_e x$. The $\log_e$ is called Naperian Logarithm after Robert Napier but is more commonly called the natural logarithm and denoted ln. In engineering, many functions are exponential and the solutions of equations are solved by taking the natural logarithm.

WORKED EXAMPLE No. 3

A well known formula used in the analysis of damped vibrations is $\frac{x_1}{x_2} = e^{\frac{2\pi\delta}{\sqrt{1-\delta^2}}}$

Where x_1 and x_2 are the amplitude of two successive vibrations and δ is the damping ratio. Calculate δ when x_1= 3 mm and x_2 = 0.5 mm respectively. Calculate the amplitude reduction factor and the damping ratio.

SOLUTION

Take natural logarithms so that $\ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\delta}{\sqrt{1-\delta^2}}$

Now we can put in the numerical data and solve as follows.

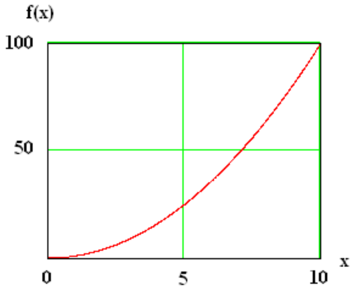
$$\ln\left(\frac{x_1}{x_2}\right) = \ln\left(\frac{3}{0.5}\right) = \ln 6 = 1.792$$

$$1.792 = \frac{2\pi\delta}{\sqrt{1-\delta^2}} \quad \text{square both sides}$$

$$3.21 = \frac{39.478\delta^2}{1-\delta^2} \quad \text{so} \quad 1-\delta^2 = 12.298\delta^2$$

$$13.298\delta^2 = 1 \text{ and } \delta^2 = \frac{1}{13.298} = 0.075 \text{ and } \delta = \sqrt{0.075} = 0.274$$

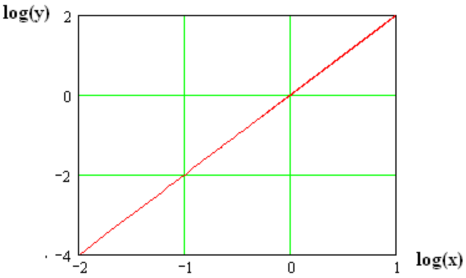
4. LINEARISING EQUATIONS WITH LOGARITHMS



Many relationships between two variable quantities are non-linear. This means that the graph of one plotted against the other is not a straight line. For example consider the simple equation:

$$f(x) = y = x^2$$

If we plot them we get the curved graph shown.



It is possible to represent a formula like this in a form that produces a straight line graph and the simplest way is use logarithms. If we take logs we get: log(y) = 2 log x
Plotting this we get a straight line graph as shown. Note that the gradient is 2.

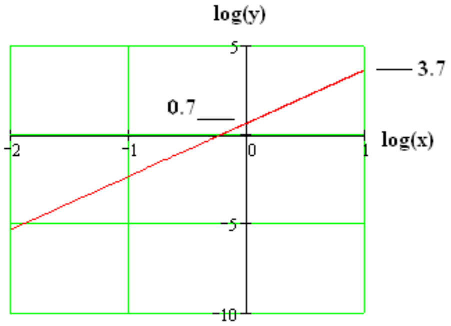
The general equation $y = C x^n$ may be reduced to a straight line in the same way. Taking logs we have:

$$\log(y) = \log(C) + n\log(x)$$

This gives a straight line with a gradient n and an intercept C

WORKED EXAMPLE No. 4

Deduce the relationship between y and x from the graph shown.



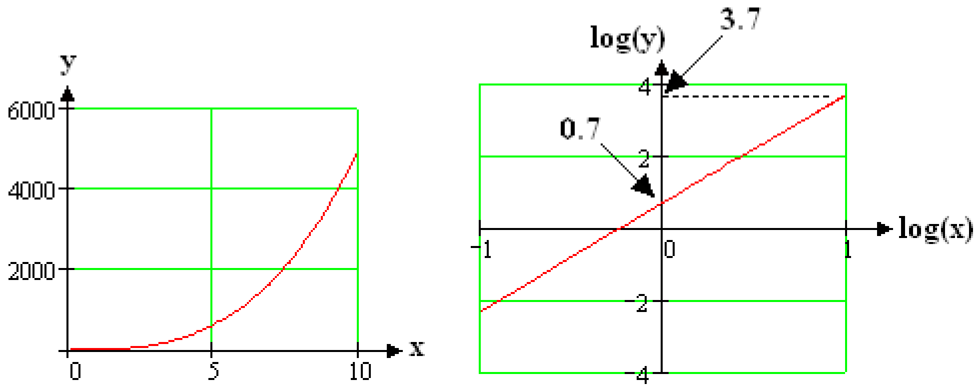
SOLUTIONS

The intercept is 0.7 so log(C) = 0.7. Find the antilog and C = 5
The gradient is (3.7 – 0.7)/1 = 3 so n = 3

The equation is $y = 5x^3$

WORKED EXAMPLE No. 5

The graph shows the results of an experiment in which a variables x and y are recorded and plotted. When log(x) and log(y) are plotted the straight line graph shown is produced. Determine the function f(x).



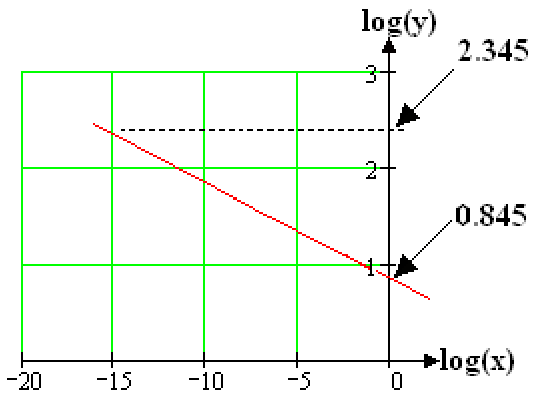
SOLUTION

From the straight line graph we have an intercept of 0.7 and a gradient of (3.7 – 0.7)/1 = 3

$\varphi(x) = 0.7 + 3 \log(x)$ Take antilogs $f(x) = 5x^3$

WORKED EXAMPLE No. 6

The graph shows the results of an experiment in which a variables x and y are recorded and plotted as logs. Determine the function f(x).



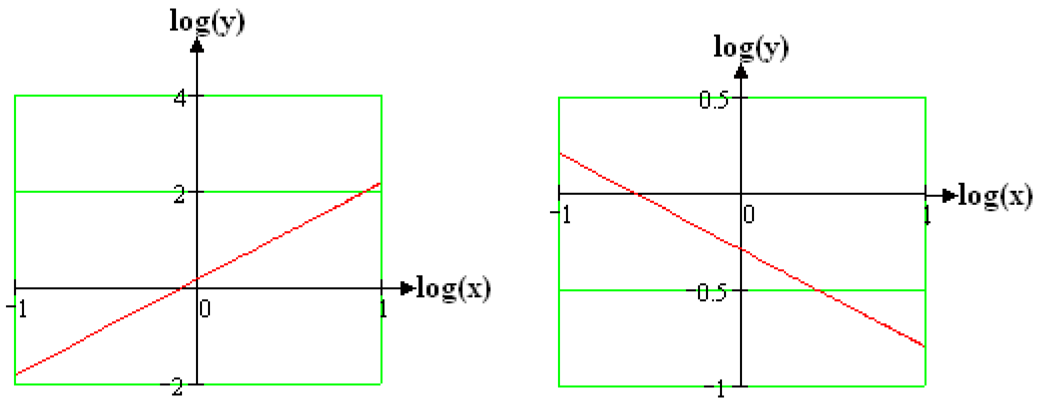
SOLUTION

From the straight line graph we have an intercept of 0.845.
The gradient is (2.345– 0.845)/(-15) = - 0.1

$\varphi(x) = 0.845 - 0.1 \log(x)$ Take antilogs $f(x) = 7x^{-0.1}$

SELF ASSESSMENT EXERCISE No. 1

1. Determine the function f(x) for each of the graphs below,



Answers $f(x) = 1.5x^2$ and $f(x) = 0.5x^{-0.5}$

2. When a pulley belt slips on a wheel it is believed the largest force F_1 on one side and the smaller force F_2 on the other side is in the form:

$$F_1 = F_2 e^{\mu\theta}$$

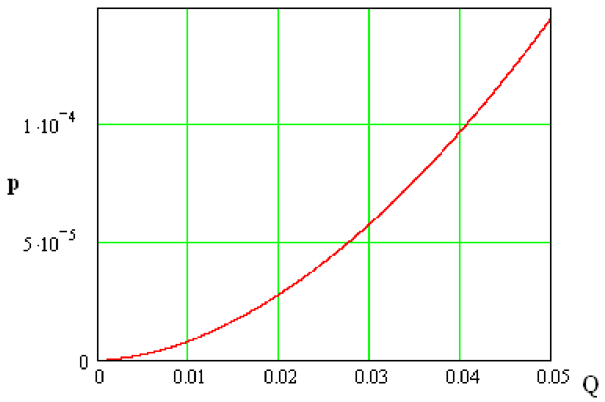
μ is the coefficient of friction which is constant and unknown. θ is the angle of contact.

If an experiment is conducted with various forces and angles to verify this relationship and the only unknown is μ, how should the data be plotted so that a straight results if the equation is true? How may the value of μ be found from the straight line graph?

3. The pressure loss in a given pipe is believed to be related to the flow rate by the formula:

$$p = kQ^n$$

An experiment is conducted and the pressure is measured over a range of flow rates. The results and plot are as shown.



p (Pa)	Q (m^3/s)
$127.394 \cdot 10^{-9}$	$1 \cdot 10^{-3}$
$264.31 \cdot 10^{-9}$	$1.5 \cdot 10^{-3}$
$443.613 \cdot 10^{-9}$	$2 \cdot 10^{-3}$
$662.891 \cdot 10^{-9}$	$2.5 \cdot 10^{-3}$
$920.382 \cdot 10^{-9}$	$3 \cdot 10^{-3}$
$1.215 \cdot 10^{-6}$	$3.5 \cdot 10^{-3}$
$1.545 \cdot 10^{-6}$	$4 \cdot 10^{-3}$
$1.91 \cdot 10^{-6}$	$4.5 \cdot 10^{-3}$
$2.308 \cdot 10^{-6}$	$5 \cdot 10^{-3}$
$2.74 \cdot 10^{-6}$	$5.5 \cdot 10^{-3}$
$3.205 \cdot 10^{-6}$	$6 \cdot 10^{-3}$
$3.702 \cdot 10^{-6}$	$6.5 \cdot 10^{-3}$
$4.23 \cdot 10^{-6}$	$7 \cdot 10^{-3}$
$4.789 \cdot 10^{-6}$	$7.5 \cdot 10^{-3}$
$5.379 \cdot 10^{-6}$	$8 \cdot 10^{-3}$
$5.999 \cdot 10^{-6}$	$8.5 \cdot 10^{-3}$

Process the data and plot it so that a straight line is produced and deduce the values of k and n.
(Answers 0.032 and 1.8 note that you will have to project the graph to find the intercept)

4. When a capacitor C discharges through a resistor R the voltage V at any time t after the start is related to time t by $V = 12(1 - e^{-t/RC})$. Make t the subject of the formula. Given RC = 2 seconds evaluate the time when V = 6 Volts.

(Answers $t = -RC \ln(1 - V/12)$ and 1.386 s)

5. HYPERBOLIC FUNCTIONS

These are functions that seem to resemble trigonometric functions. They are used in the solution of mathematical problems. They should be remembered from their basic definitions which are as follows.

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \text{ and } \cosh(x) = \frac{e^x + e^{-x}}{2} \text{ and } \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

These are pronounced shine, cosh and than respectively.

We can use the expansion for e^x to show that:

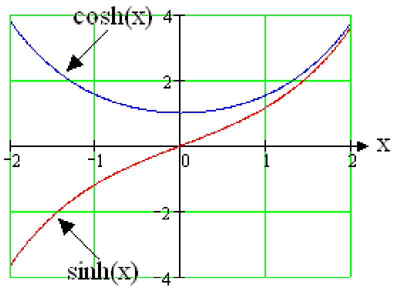
$$\sinh(x) = \frac{\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right]\ldots\ldots\ldots - \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\ldots\ldots\right]}{2} = \frac{2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \ldots\ldots}{2}$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots$$

$$\cosh(x) = \frac{\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right]\ldots\ldots\ldots + \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\ldots\ldots\right]}{2} = \frac{2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \ldots\ldots}{2}$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots$$

The graphs of the two functions are shown.



Further work would show that many hyperbolic functions are similar to trigonometric functions and the table shows this.

Hyperbolic function	**Trigonometric function**
$\cosh(x) + \sinh(x) = e^x$	
$\cosh(x) - \sinh(x) = e^{-x}$	
$y = \sinh(x)$ then $\frac{dy}{dx} = \cosh(x)$	$y = \sin(x)$ then $\frac{dy}{dx} = \cos(x)$
If $y = \cosh(x)$ then $\frac{dy}{dx} = \sinh(x)$	If $y = \cos(x)$ then $\frac{dy}{dx} = -\sin(x)$
$\int \sinh(x)dx = \cosh(x) + C$	$\int \sin(x)dx = -\cos(x) + C$
$\int \cosh(x)dx = \sinh(x) + C$	$\int \cos(x)dx = \sin(x) + C$
$\cosh^2(x) - \sinh^2(x) = 1$	$\cos^2(x) + \sin^2(x) = 1$
$\sinh(A \pm B) = \sinh(A)\cosh(B) \pm \cosh(A)\sinh(B)$	$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
$\cosh(A \pm B) = \cosh(A)\cosh(B) \pm \sinh(A)\sinh(B)$	$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$

WORKED EXAMPLE No. 7

Using the definition $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$ evaluate cosh(3) to 2 decimal places.

SOLUTION

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... \quad = 1 + \frac{3^2}{2!} + \frac{3^4}{4!} + \frac{3^6}{6!} + \frac{3^8}{8!} + \frac{3^{10}}{10!}. + \frac{3^{12}}{12!}..$$

$$= 1 + 4.5 + 3.375 + 1.0125 + 0.1627 + 0.0163 + 0.0011$$

$$= 10.07$$

WORKED EXAMPLE No. 8

Show from first principles that sinh(A+B) = sinh(A) cosh(B) + cosh(A) sinh(B)

SOLUTION

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \text{ and } \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sinh(A)\cosh(B) + \cosh(A)\sinh(B) = \left[\frac{e^A - e^{-A}}{2}\right]\left[\frac{e^B + e^{-B}}{2}\right] + \left[\frac{e^A + e^{-A}}{2}\right]\left[\frac{e^B - e^{-B}}{2}\right]$$

$$= \left[\frac{e^A e^B + e^A e^{-B}}{4} - \frac{e^{-A} e^B + e^{-A} e^{-B}}{4}\right] + \left[\frac{e^A e^B - e^A e^{-B}}{4} + \frac{e^{-A} e^B - e^{-A} e^{-B}}{4}\right]$$

$$= \left[\frac{e^A e^B + e^A e^{-B}}{2}\right] = \left[\frac{e^{A+B} + e^{A-B}}{2}\right] = \left[\frac{e^{(A+B)} + e^{-(A+B)}}{2}\right] = \sinh(A+B)$$

SELF ASSESSMENT EXERCISE No. 2

1. Show from first principles that cosh(A+B) = cosh(A) cosh(B) + sinh(A) sinh(B)

2. Using the definition $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$ evaluate sinh(1) to 3 decimal places. (1.175)

3. Using the definition $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$ evaluate cosh (2) to 3 decimal places. (3.762)