EDEXCEL NATIONAL CERTIFICATE/DIPLOMA

ADVANCED MECHANICAL PRINCIPLES AND APPLICATIONS
UNIT 18
NQF LEVEL 3

OUTCOME 2
BE ABLE TO DETERMINE THE STRESS DUE TO BENDING IN BEAMS AND TORSION IN POWER TRANSMISSION SHAFTS

TUTORIAL 2 - BENDING STRESS IN BEAMS

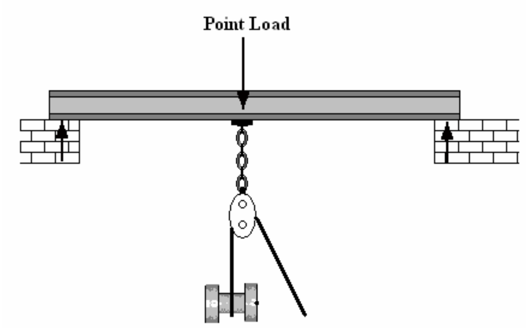
Direct stress due to bending: expressions for second moment of area of solid and hollow rectangular and circular beam sections; application of bending equation ($\sigma/y = M/I = E/R$) to determine stress due to bending and radius of curvature at a beam section; determination of factor of safety in operation

Shear stress due to torsion: expressions for polar second moment of area of solid and hollow circular transmission shaft sections; application of torsion equation ($\tau/r = T/J = G\theta/l$) and expression for power transmitted ($Power = T\omega$) to determine induced shear stress and angle of twist; determination of factor of safety in operation

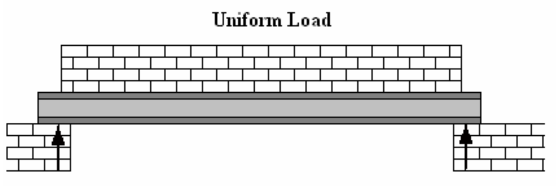
You should already know how to determine the bending moment in simple beams, if not, you need to study tutorials on bending moments. You should also have studies second moments of areas for simple shapes in tutorial 1.

Let's start by revising the definition of a beam.

1. REVISION OF BEAMS AND LOADS

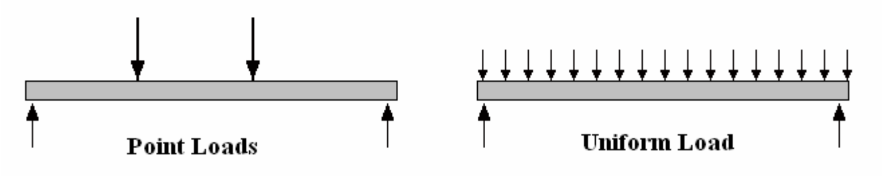


A beam is a structure that is loaded laterally (sideways) to its length. These loads produce bending and bending is the most severe way of stressing a component. Suppose you were given a simple rod or a ruler and asked to break it. You would struggle to break it by stretching it or twisting it but it would be easily to break it by bending it.

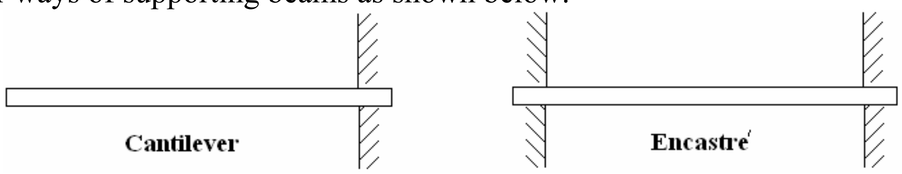


A beam may have point loads. It may also have a uniformly distributed load (udl) such as might occur due to its weight or the weight of a wall built along it.

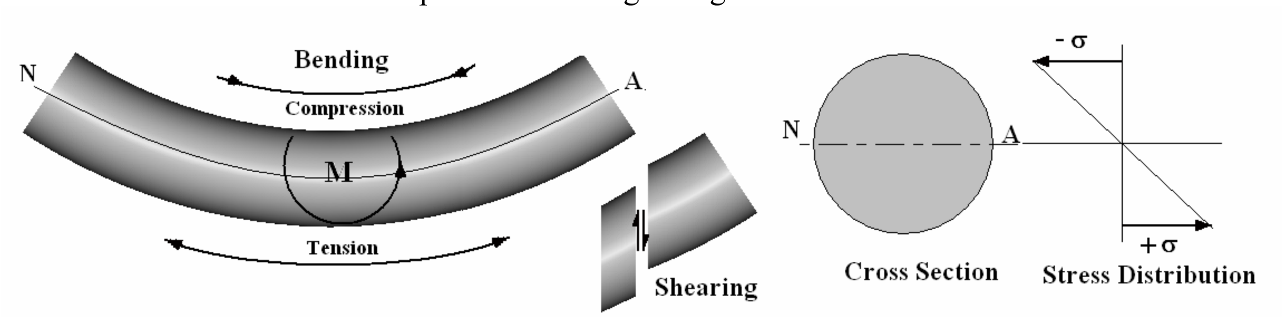
We normally show the loads with simplified diagrams like this.



There are other ways of supporting beams as shown below.



The loads produce shear force and bending moments that vary with position along the length. You should already know how to construct shear force and bending moment diagrams for simply supported beams. When a bending moment M is applied to a beam, one surface is compressed (negative stress) and the other is stretched (tensile positive stress). The stress varies across the section from a maximum negative to a maximum positive as shown. Somewhere in between there is a longitudinal layer that is not stressed (neutral) and this layer lays on the NEUTRAL AXIS. The neutral axis is through the centroid for pure bending and for symmetrical sections this is the middle. In addition the transverse forces produce shearing on a given section as indicated.



The purpose of this tutorial is to enable you to calculate the stress due to the bending moment. We will derive the following three part equation known as the bending equation.

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

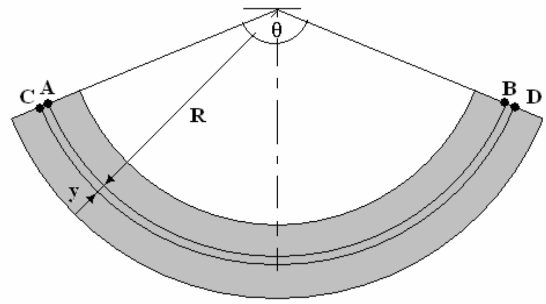
M is the bending moment at a given point along the length.
I is the second moment of area of the sectional area about the neutral axis.
σ is the stress due to bending at a distance y from the neutral axis.
E is the modulus of elasticity.
R is the radius of curvature.

2. THE BENDING FORMULA

2.1 NEUTRAL AXIS

This is the axis along the length of the beam which remains unstressed, neither compressed nor stretched when it is bent. Normally the neutral axis passes through the centroid of the cross sectional area. For simple rectangular and circular sections, this is the axis along the centre line.

Consider that the beam is bent into an arc of a circle through angle θ radians. AB is on the neutral axis and is the same length before and after bending. The radius of the neutral axis is R.



Remember the length of an arc is radius x angle in radians

2.2 RADIUS OF CURVATURE

Normally the beam does not bend into a circular arc. However, what ever shape the beam takes under the sideways loads; it will basically form a curve on an x – y graph. In maths, the radius of curvature at any point on a graph is the radius of a circle that just touches the graph and has the same tangent at that point.

2.3 RELATIONSHIP BETWEEN STRAIN AND RADIUS OF CURVATURE

The length of AB $AB = R\,\theta$

Consider a layer of material distance y from the neutral axis as shown. This layer is stretched because it must become longer and the material has stress and strain in it in a lengthwise direction as a result. (If y was to the inside of the neutral axis it would be compressed and become shorter).

The radius of the layer is R + y.

The length of this layer is the line DC. $DC = (R + y)\theta$

This layer is strained and strain (ε) is defined as ε = change in length/original length

Substitute $AB = R\,\theta$ and $DC = (R + y)\theta$

$$\varepsilon = \frac{DC - AB}{AB} = \frac{(R + y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

The modulus of Elasticity (E) relates direct stress (σ) and direct strain (ε) for an elastic material and the definition is as follows.

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma}{\varepsilon}$$

Substitute $\varepsilon = \frac{y}{R}$ and $E = \frac{\sigma R}{y}$ $\qquad \frac{E}{R} = \frac{\sigma}{y}$

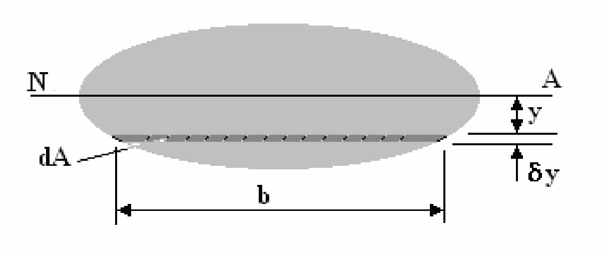
It follows that stress and strain vary along the length of the beam depending on the radius of curvature.

We will now go on to show that the radius of curvature depends upon the bending moment M acting at any given point along the length of the beam.

2.4 RELATIONSHIP BETWEEN STRESS AND BENDING MOMENT

This section is more difficult to understand but necessary if you are to understand the importance of the second moment of area on the stress in the beam. Consider a beam with a uniform section along its length. An arbitrary oval shape is shown here. Think of the beam as being made of many thin layers of material running the length of the beam and held together by molecular forces.

Consider one such elementary layer at a given point along the length at a distance y from the neutral axis. When the cross section is viewed end on it appears as an elementary strip width b and thickness δy.



The cross sectional area is A. The elementary strip is a small part of the total cross sectional Area and is denoted in calculus form as δA. The strip may be regarded as a thin rectangle width b and height δy so

$$\delta A = b\,\delta y$$

The stress on the strip is $\sigma = Ey/R$

If the layer shown is stretched, then there is a small force δF pulling normal to the section trying to slide the layer out of the material in a lengthwise direction. This force must be the product of the stress and the area and is a small part of the total force acting on the section δF.

$$\delta F = \sigma\,\delta A \quad \text{Substitute } \sigma = \frac{Ey}{R} \text{ and } \delta F = \frac{Ey}{R}\delta A$$

Consider that the whole beam is made up of many such layers. Some are being stretched and pull normal to the section and some are compressed and push. The total force acting on the section is the sum of all these small forces.

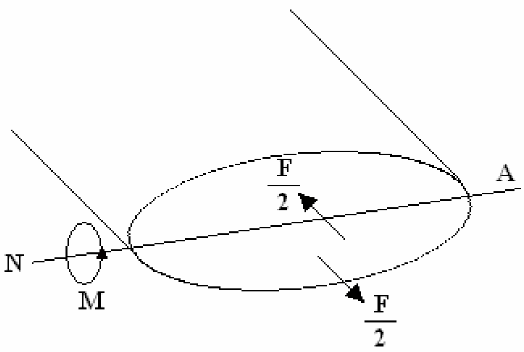
$$F = \sum \delta F = \sum \frac{Ey}{R}\delta A$$

In the limit as δy tends to zero, the number of strips to be summed tends to infinity. The small quantities δy and δA become the differential coefficient dy and dA. The total force is given by the integration $F = \int_{bottom}^{top} \frac{Ey}{R} dA = \frac{E}{R}\int_{bottom}^{top} y\,dA$

The expression $\int_{bottom}^{top} y\,dA$ is by definition the first moment of area of the shape about the centroid.

Evaluating this expression would give zero since any first moment of area is zero about the centroid.

The centroid in this case is on the neutral axis. The areas above and below the neutral axis are equal. Half the force is a compressive force pushing into the diagram, and half is tensile pulling out. They are equal and opposite so it follows that F = 0 which is sensible since cross sections along the length of a beam obviously are held in equilibrium.



The diagram indicates that the two forces produce a turning moment about the neutral axis because half the section is in tension and half in compression. This moment must be produced by the external forces acting on the beam making it bend in the first place. This moment is called the bending moment M and this is the same bending moment found by calculation using the external loads.

Consider the moment produced by the force on the elementary strip δF. It acts a distance y from the neutral axis so the moment produced is δM = y δF

In the limit as δy tends to zero the total moment is found by reverting to calculus again.

$$M = \sum y\delta\delta = \int_{bottom}^{top} y\,dF = \int_{bottom}^{top} y\frac{Ey}{R}\,dA$$

$$M = \frac{E}{R}\int_{bottom}^{top} y^2\,dA$$

The expression $\int_{bottom}^{top} y^2\,dA$ is by definition the **second moment of area** about the neutral axis and this is not zero but has a definite value. In general it is denoted by the symbol **I**.

$$I = \int_{bottom}^{top} y^2\,dA$$

We may now write the moment as $M = \frac{E}{R}I$ and rearrange it to $\frac{M}{I} = \frac{E}{R}$

Combining $\frac{E}{R} = \frac{\sigma}{y}$ and $\frac{M}{I} = \frac{E}{R}$ we now have $\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$

This is called the bending equation and it has 3 parts.

If the stress is required at a given point along the beam we use either $\sigma = \frac{My}{I}$ or $\sigma = \frac{Ey}{R}$

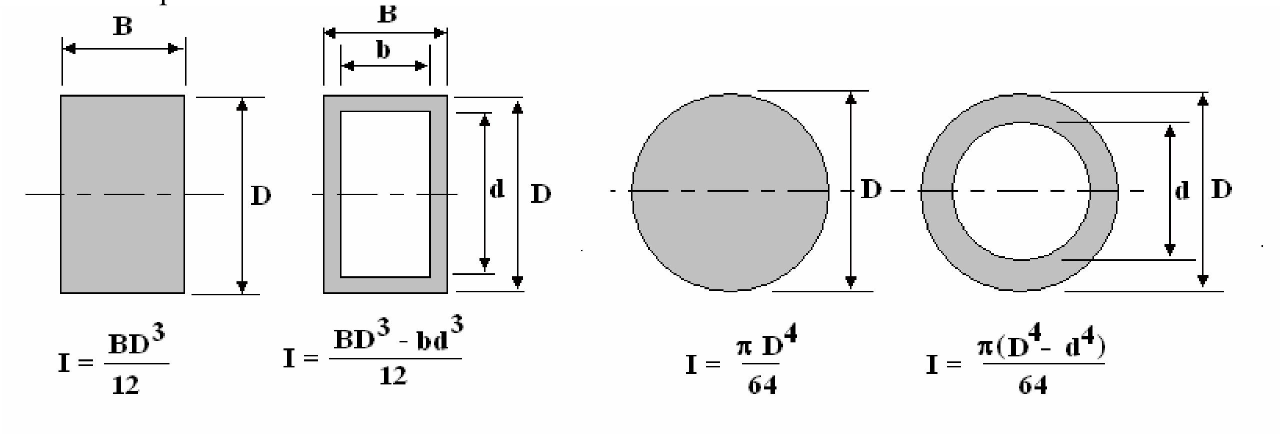
This indicates that the stress in a beam depends on the bending moment and so the maximum stress will occur where the bending moment is a maximum along the length of the beam. It also indicates that stress is related to distance y from the neutral axis so it varies from zero to a maximum at the top or bottom of the section. One edge of the beam will be in maximum tension and the other in maximum compression. In beam problems, we must be able to deduce the position of greatest bending moment along the length.

2.5 STANDARD SECTIONS

For a given section the value of I may be determined by mathematics. Many beams are manufactured with standard sections. British Standard BS4 part 1 gives the properties of standard steel beams and joists. The areas and second moments of area are listed in the standards and since the distance y to the edge is also known they list a property called the ELASTIC MODULUS and this is defined as $z = \frac{I}{y}$. The stress at the edge of the beam is then found from the equation:

$$\sigma = \frac{My}{I} = \frac{M}{Z}$$

For standard shapes the second moment of area can be deduced. The following formulae apply to standard shapes.



For more complex shapes such as TEE and U sections, you will need further studies.

WORKED EXAMPLE No. 1

A beam has a rectangular cross section 80 mm wide and 120 mm deep. It is subjected to a bending moment of 15 kNm at a certain point along its length. It is made from metal with a modulus of elasticity of 180 GPa. Calculate the maximum stress on the section.

SOLUTION

B = 80 mm, D = 100 mm. It follows that the value of y that gives the maximum stress is 50 mm. Remember all quantities must be changed to metres in the final calculation.

$$I = \frac{BD^3}{12} = \frac{80 \times 100^3}{12} = 6.667 \times 10^6 \text{ mm}^4 = 6.667 \times 10^{-6} \text{ m}^4$$

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\sigma = \frac{My}{I} = \frac{15 \times 10^3 \times 0.05}{6.667 \times 10^{-6}} = 112.5 \times 10^6 \text{ N/m}^2$$

WORKED EXAMPLE No. 2

A beam has a hollow circular cross section 40 mm outer diameter and 30 mm inner diameter. It is made from metal with a modulus of elasticity of 205 GPa. The maximum tensile stress in the beam must not exceed 350 MPa.

Calculate the following.

(i) the maximum allowable bending moment.
(ii) the radius of curvature.

SOLUTION

D = 40 mm, d = 30 mm

I = $\pi(40^4 - 30^4)/64 = 85.9 \times 10^3$ mm^4 or 85.9×10^{-9} m^4.

The maximum value of y is D/2 so y = 20 mm or 0.02 m

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$M = \frac{\sigma I}{y} = \frac{350 \times 10^6 \times 85.9 \times 10^{-9}}{0.02} = 1503 \text{ Nm or } 1.503 \text{ MNm}$$

$$\frac{\sigma}{y} = \frac{E}{R}$$

$$R = \frac{Ey}{\sigma} = \frac{205 \times 10^9 \times 0.02}{350 \times 10^6} = 11.71 \text{ m}$$

SELF ASSESSMENT EXERCISE No.1

1. A beam has a bending moment (M) of 3 kNm applied to a section with a second moment of area (**I**) of 5×10^{-3} m^4. The modulus of elasticity for the material (E) is 200×10^9 N/m^2. Calculate the radius of curvature. (Answer 333.3 km).

2. The beam is Q1 has a distance from the neutral axis to the edge in tension of 60 mm. Calculate the stress on the edge. (Answer 36 kPa).

3. A beam under test has a measured radius of curvature of 300 m. The bending moment applied to it is 8 Nm. The second moment of area is 8000 mm^4. Calculate the modulus of elasticity for the material. (Answer 300 GPa).

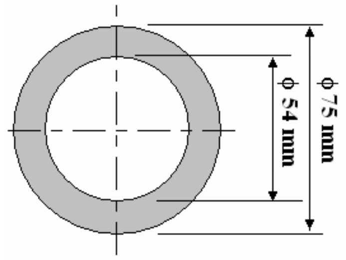
4. A beam is made from round tube 120 mm outer diameter and 100 mm inner diameter. If the bending moment at a given point is 6 kNm determine the stress at the outer edge and the radius of curvature. Take E = 205 GPa (68.3 MPa and 180 m)

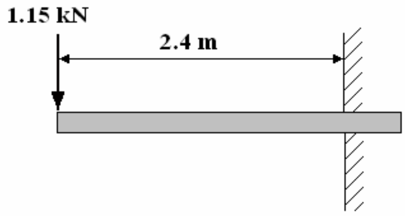
3. SAFETY FACTOR

It is normal to design any structure so that the stress does not exceed the maximum allowed. This might be the yield stress or the ultimate stress or some other figure. The safety factor is defined as the ratio of the maximum allowable stress to the actual stress.

WORKED EXAMPLE No. 3

A simple cantilever has a single point load at the end as shown and a cross section as shown. The maximum allowable tensile stress is120 MPa. Calculate the safety factor.





SOLUTION

First it should be fairly obvious that the maximum bending moment in the beam is at the wall and is :   M = 1150 x 2.4 = 2760 N m.

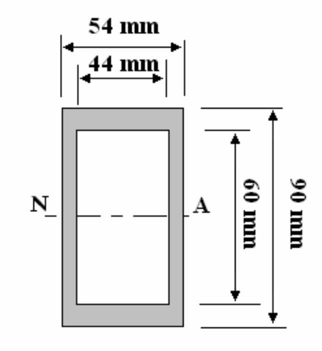
Next calculate the second moment of area. $I = \frac{\pi\left(0.075^4 - 0.054^4\right)}{64} = 1.136 \times 10^{-6}\,\text{m}^4$

Next calculate the stress. $\frac{M}{I} = \frac{\sigma}{y}$ $\quad \sigma = \frac{My}{I} = \frac{2760 \times 0.0375}{1.136 \times 10^{-6}} = 91.128\ \text{MPa}$

The safety factor is 120/91.128 = 1.317

WORKED EXAMPLE No. 4

A beam is made from a hollow box section as shown. The maximum allowable stress is 450 MPa. Calculate the maximum allowable bending moment about the neutral axis (NA) if a safety factor of 3 is to be used.



SOLUTION

First calculate I $\quad I = \frac{BD^3 - bd^3}{12} = \frac{54 \times 90^3 - 44 \times 60^3}{12} = 2.489 \times 10^6\,\text{mm}^4 = 2.489\text{x}10^{-6}\,\text{m}^4$

The working stress is 450/3 = 150 MPa y = 90/2 = 45 mm

Next calculate the bending moment. $\quad M = \frac{\sigma I}{y} = \frac{150 \times 10^6 \times 2.489 \times 10^{-6}}{0.045} = 8.295\ \text{kNm}$

SELF ASSESSMENT EXERCISE No. 2

1. A vertical pole is made of alloy tube 50 mm outer diameter and 38 mm inner diameter. It is placed in a ground socket and is free standing. If the yield stress is 550 MPa determine the bending moment that will make it fail. If the safety factor to be used is 2.0 determine the maximum bending moment allowed. (4.5 kNm and 2.225 kNm)

2. A beam is made from a hollow box section 50 mm x 75 mm outer dimensions and 40 mm x 55 mm inner dimensions. The maximum allowable stress is 300 MPa. Calculate the maximum allowable bending moment about the neutral axis parallel with the shorter edge. A safety factor of 2 is to be used. The beam is symmetrical about the centre line. (4.813 kNm)